Portfolio insurance: Gap risk under conditional multiples

Hachmi BEN AMEUR*; Jean-Luc PRIGENT**
*INSEEC, Paris
**THEMA, University of Cergy-Pontoise, 33 bd du Port, 95011, Cergy, France. Email: jean-luc.prigent@u-cergy.fr

Bogota seminar December 2012
Presentation schedule

- Introduction about portfolio insurance
- The standard CPPI method
- The CPPI method with a conditional multiple
- Quantile and expected shortfall conditions
- Numerical illustrations
Portfolio insurance has two main objectives:
- First, to allow investors to recover at maturity at least a given percentage of his initial investment, usually 100% (it limits downside risk in bearish financial market, dramatically relevant during financial crisis).
- Second, to allow investors to benefit from potential market rises (it allows investors to benefit from bullish markets).

The two main portfolio insurance methods:
- The **Option Based Portfolio Insurance (OBPI)**, introduced by Leland and Rubinstein (1976): the portfolio is invested in a risky benchmark asset covered by a put option written on it. The strike of the option is equal to a fixed proportion of the initially invested amount (which corresponds to the capital insured at maturity).
- The **Constant Proportion Portfolio Insurance (CPPI)** considered by Perold (1986) for fixed-income instruments and Black and Jones (1987) for equity instruments (see also Perold and Sharpe, 1988; Black and Perold, 1992).
In France, according to the AMF, the first method corresponds to structured portfolio management (about 700 "fonds à formule").

The CPPI method belongs to the family of "fonds diversifiés".

**Graphique 3** – Évolution des encours des OPCVM à vocation générale (hors OPCVM nourricier: entre 2008, 2009 et 2010 répartis par catégorie (exprimés en milliards d’euros)
The standard CPPI method (problem)
The standard CPPI method (asset dynamics)

Two basic assets: a money market account $B$, and a portfolio of traded assets such as a composite index, denoted by $S$. Time period $[0, T]$. Self-financing strategies.

Riskless asset $B$ ($r$ deterministic interest rate): $dB_t = B_t r dt$,

Risky asset price $S$ (diffusion process with jumps):

$$dS_t = S_t [\mu(t, S_t) dt + \sigma(t, S_t) dW_t + \delta(t, S_t) dN_t],$$

where $(W_t)_t$ is a standard Brownian motion, independent of the Poisson process having the jump measure $N$.

Remark

The random variables $(\Theta_{n+1} - \Theta_n)$, which represent the times between two consecutive jumps, are independent and have the same exponential probability distribution with parameter denoted by $\lambda$. The relative jumps of the risky asset $\frac{\Delta S_{\Theta_n}}{S_{\Theta_n}}$ are equal to $\delta(\Theta_n, S_{\Theta_n})$ (strictly greater than $-1$ to get the positivity of the asset $S$).
The standard CPPI method (strategy)

- The floor $F$ at any time $t$ of the management period (dynamic insured amount):
  \[ dF_t = F_t r dt, \text{ with } F_0 = pe^{-rT} V_0. \]

- The difference $(V_0^{CPPI} - F_0)$ is called the cushion. It is denoted by $C_0$. Its value $C_t$ at any time $t$ in $[0, T]$ is given by:
  \[ C_t = V_t^{CPPI} - F_t. \]

- Denote by $e_t$ the exposure. It is the total amount invested in the risky asset.

- The standard CPPI method consists of letting $e_t = mC_t$ where $m$ is a constant called the multiple. The interesting case is when $m > 1$, that is, when the portfolio profile is convex. Thus, the CPPI method is parametrized by $F_0$ and $m$. Note that the multiple must not be too high as shown for example in Prigent (2001) and Bertrand and Prigent (2002).
The standard CPPI method (general formula)

Proposition

(See Prigent (2001) The cushion value $C_t$ at any time is given by:

$$C_0 \exp \left[ (1 - m)rt + m \int_0^t \left( \mu - \frac{1}{2} m \sigma^2(s, S_s) \right) ds + m \int_0^t \sigma(s, S_s) dW_s \right]$$

$$\times \prod_{0 \leq \Theta_n \leq t} \left( 1 + m \delta(\Theta_n, S_{\Theta_n}) \right).$$

Corollary

The guarantee is satisfied if and only if the relative jumps of asset $S$ satisfy:

$$\delta(\Theta_n, S_{\Theta_n}) \geq -1/m.$$

Thus, when the jumps are smaller than a negative constant $d$, then condition $0 \leq m \leq -1/d$ allows to get the guarantee. For example if $d$ is equal to $-10\%$, then $m \leq 10$. This condition depends neither on the law of jump times $\Theta_n$, nor on the whole probability distribution of the jumps.
Corollary

The cushion value is given by: \( C_t = C_0 e^{m \sigma W_t + [r + m(\mu - r) - \frac{m^2 \sigma^2}{2}]t} \) with \( C_0 = V_0 - F_0 \).

The value \( V_{t_{CPPI}} \) of the portfolio is given by:

\[
V_{t_{CPPI}}(m, S_t) = F_0 e^{rt} + \alpha_t S_t^m,
\]

where \( \alpha_t = \left( \frac{C_0}{S_0^m} \right) \exp[\beta t] \), and \( \beta = \left( r - m (r - \frac{1}{2} \sigma^2) - \frac{m^2 \sigma^2}{2} \right) \).

Corollary

- The insurance is perfect and the cushion value and the portfolio value are independent of the risky asset paths.
- Their probability distributions are lognormal (up to a translation for the portfolio value) with a volatility equal to \( m \sigma \). The instantaneous mean rate of return is equal to \( r + m(\mu - r) \). The multiple \( m \) can be viewed as a weight in the volatility and in the excess return \( (\mu - r) \).
Corollary

(Compound Poisson process case) Assume that the relative jumps of asset \( S \) are independent and identically distributed with probability distribution \( K(dx) \). Denote by \( b \) their common mean \( \mathbb{E}[\delta(|\Theta_1, S_{\Theta_1}|)] \), and by \( c \) their moment \( \mathbb{E}[\delta^2(|\Theta_1, S_{\Theta_1}|)] \).

Then, the portfolio value has mean and variance, respectively given by:

\[
\begin{align*}
\mathbb{E}[V_t] &= (V_0 - F_0) e^{[r + m(\mu + b \lambda - r)]t} + F_0 e^{rt}, \\
\text{Var}[V_t] &= (V_0 - F_0)^2 e^{2[r + m(\mu + b \lambda - r)]t} \left[ e^{m^2(\sigma^2 + c \lambda)t} - 1 \right].
\end{align*}
\]

No mean-variance dominance since both mean and variance are increasing functions of the multiple \( m \).

No first-order stochastic dominance (Bertrand and Prigent, 2005). See also Zagst and Kraus (2011).
The standard CPPI method (pdf)

Corollary

The cushion density at time \( t \) when there is no jump \( g_{0,t} \) is given by:

\[
g_{0,t}(x) = \frac{I_{x>0}}{x \sqrt{2\pi \sigma^2 m^2 t}} e^{-\frac{1}{2 \sigma^2 m^2 t} \left( \ln \left[ \frac{x}{C_0} \right] - t (r + m (\mu - r) - \frac{m^2 \sigma^2}{2}) \right)^2}.
\]

In the presence of jumps but for a perfect guarantee:

\[
g_{Ct}(x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \int_{\mathbb{R}^n} g_{0,t} \left( \frac{x}{\prod_{i \leq n} (1 + my_i)} \right) K^n(dy_1, \ldots, dy_n) \prod_{i \leq n} (1 + my_i),
\]

where \( K^n \) designates the joint distribution of i.i.d. variables \( \delta(\Theta_i, S_{\Theta_i}), i \leq n \): \( K^n(dy_1, \ldots, dy_n) = \otimes K(dy_i) \). Finally, the portfolio pdf is deduced:

\[
f_{Vt}(x) = g_{Ct}(x - F_0 e^{rt}).
\]
The standard CPPI method (discrete-time)

- Time periods \([t_{k-1}, t_k]\).

\[
V_{t_k} = V_{t_{k-1}} + e_{t_{k-1}} \frac{\Delta S_{t_k}}{S_{t_{k-1}}} + (V_{t_{k-1}} - e_{t_{k-1}}) r_{t_k},
\]

from which, we deduce the cushion dynamics:

\[
C_{t_k} = C_{t_{k-1}} \left[1 + m \frac{\Delta S_{t_k}}{S_{t_{k-1}}} + (1 - m) r_{t_k}\right].
\]

- Thus:

\[
V_{t_k} = F_{t_k} + C_0 \prod_{l=1}^{t_k} \left[1 + m \frac{\Delta S_{t_l}}{S_{t_{l-1}}} + (1 - m) r_{t_l}\right].
\]
Now the CPPI multiple varies at times $t_k$.
The multiple is assumed to be constant (equal to $m_{t_{k-1}}$) on the time period $[t_{k-1}, t_k]$.

**Goal:** to provide explicit conditions on the sequence of multiples $(m_{t_k})_k$ to control the gap risk due to potential sharp drops in the risky asset price $S$.

This risk can be controlled by adjusting the multiple value for example by means of a quantile criterion corresponding to a Value-at-Risk approach.

The standard geometric Brownian motion does not take account of potential jumps nor any temporal dependence of financial returns. Portfolio rebalancing is also assumed to be in continuous-time, which is not exactly the usual practice.

To overcome these drawbacks, it is possible to use discrete-time processes such as the huge family of ARCH models that take particularly the dependence between returns into account.
The CPPI method with a conditional multiple (2)

Control of the gap risk: consider a sequence of positive thresholds \( (L_{t_{k-1}})_{k} \) for potential downside cushion values. The idea is to control various probabilities that the cushions become smaller than the thresholds.

**Remark**

*(Choice of the thresholds) The value of the threshold \( L_{t_{k-1}} \) at time \( t_{k-1} \) can depend on the risk tolerance of the investor or be imposed by specific guarantee constraints. For example, we can set:

1) \( L_{t_{k-1}} = 0 \). It corresponds to the standard insurance constraint: \( C_{t_k} > 0 \), which means that the portfolio value must be above the floor.

2) \( L_{t_{k-1}} = L \) constant. We can choose for example a constant value for \( L \) as a given proportion of the initial cushion \( L = qC_{t_0} \).

3) \( L_{t_{k-1}} = qC_{t_{k-1}} \), where \( q \) is a fixed proportion of the cushion values. In that case, we search to get \( C_{t_k} > qC_{t_{k-1}} \). In this case, the threshold is adjusted to the value of the cushion at any time \( t_{k-1} \). Setting \( q > 1 \) or \( q < 1 \) depends on risk tolerance.*
Quantile and expected shortfall conditions (1)

New guarantee condition at a given probability level $1 - \epsilon$:

$(L = 0; \, m \, \text{fixed})$

$P[C_{t_k} \geq 0, \forall t_k \in [0, \, T]] \geq 1 - \epsilon.$

Let $M_T$ be the maximum of the $-\frac{\Delta S_{tk}}{S_{tk-1}}$ with inverse cdf $F_{M_T}^{-1}$. For small $r_k$, we have (approximately) the following condition:

**Proposition**

$m \leq \frac{1}{F_{M_T}^{-1}((1-\epsilon))}$ and if $(\frac{\Delta S_{tk}}{S_{tk-1}})_k$ is i.i.d. with common cdf $F$:

$$m \leq \frac{1}{F^{-1}\left((1 - \epsilon)^{\frac{1}{T}}\right)}.$$

Upper limit on the multiple $m$ obviously higher than the standard limit $\frac{1}{d}$. For continuous-time and Lévy case: Prigent (2001); Cont and Tankov (2009).
Quantile and expected shortfall conditions (2)

First global quantile condition: (any \(L; m\) variable)

\[
P[\forall t_k, C_{t_k} > L_{t_{k-1}}] \geq 1 - \epsilon.
\]

This condition can be deduced from other mild assumptions:

1. Assumption (A1):
\[
\forall k, P[C_{t_k} > L_{t_{k-1}} \mid C_{t_1} > L_0, \ldots, C_{t_{k-1}} > L_{t_{k-2}}] \geq (1 - \epsilon)^{1/T}.
\]

2. Assumption (A2):
\[
\forall k, P^{\mathcal{F}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1 - \epsilon)^{1/T}.
\]

3. Assumption (A3):
\[
\forall k, P^{\mathcal{G}_{t_{k-1}}}[C_{t_k} > L_{t_{k-1}}] \geq (1 - \epsilon)^{1/T},
\]

with \(\mathcal{F}_{t_{k-1}}\) \(\sigma\)-algebra generated by the random variables \(X_{t_1}, \ldots, X_{t_k}\) and \(\mathcal{G}_{t_{k-1}}\) \(\sigma\)-algebra generated by \(\mathcal{F}_{t_{k-1}}\) and the random event \(\{C_{t_{k-1}} > 0\}\). Each of these three assumptions implies the global quantile condition.
Quantile and expected shortfall conditions (3)

- We focus on Condition $(A_3)$: $\mathbb{P}^{G_{t_{k-1}}}[C_t > L_{t_{k-1}}] \geq (1 - \epsilon)^{1/T}$.
- The guarantee condition at any time $t_k$ is: $C_{t_k} > L_{t_{k-1}}(C_{t_{k-1}} > 0)$.

$$C_{t_k} > L_{t_{k-1}} \iff 1 + m_{t_{k-1}} \times \frac{\Delta S_{t_k}}{S_{t_{k-1}}} > \frac{L_{t_{k-1}}}{C_{t_{k-1}}}.$$ 

If $C_{t_{k-1}} \leq 0$, then, for all the remaining time periods, the whole portfolio value is invested on the riskless asset.

- The value of the multiple $m_{t_{k-1}}$ can be searched as:

$$m_{t_{k-1}} = g_{t_{k-1}} \times \mathbb{I}_{C_{t_{k-1}} > 0} + h_{t_{k-1}} \times \mathbb{I}_{C_{t_{k-1}} < 0},$$  \hspace{1cm} (1)$$

where both $g_{t_{k-1}}$ and $h_{t_{k-1}}$ are $\mathcal{F}_{t_{k-1}}$-measurable. Since we must stop the investment on the risky asset as soon as the cushion is null, we must set $h_{t_{k-1}} = 0$. 
Quantile and expected shortfall conditions (4)

We search for explicit forms of the random variables $g_{t_{k-1}}$. We assume for example that the risky asset logreturn $Y$ follows a Garch($p$, $q$) model. The GARCH model is defined as follows. The logreturn $Y$ is defined by:

$$Y_{t_k} = \ln \left( \frac{S_{t_k}}{S_{t_{k-1}}} \right) \iff \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} = \exp(Y_{t_k}) - 1.$$ 

Consider the system of auto regressive equations:

$$\begin{cases} 
Y_{t_k} = \alpha_0 + \sum_{i=1}^{p} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times \epsilon_{t_k}, \\
\Lambda(\sigma_{t_k}) = c_0 + \sum_{j=1}^{q} c_j(\epsilon_{t_{k-j}}) \times d_1 \Lambda(\sigma_{t_{k-1}}), 
\end{cases} \tag{2}$$

where $\sigma_{t_k}$ denotes the volatility, the sequence $(\epsilon_{t_k})_k$ is i.i.d with common pdf $f_{\epsilon} > 0$ and $\Lambda$, $c_0$ and $d_1$ are constant, and $c_j(\cdot)$ are deterministic functions. The function $\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}$ is assumed to be strictly increasing. Note that the volatility $\sigma_{t_k}$ is known at time $t_{k-1}$. It means that the sequence $(\sigma_{t_k})_k$ is predictable.
The information delivered by the observation of risk asset returns until time $t_{k-1}$ is generated by the random vector $(\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}})_k$. We have:

$$\mathcal{F}_{t_{k-1}} = \sigma - \text{algebra } (\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}}).$$

Thus, the random variables $g_{t_{k-1}}$ are deterministic functions of $(\varepsilon_{t_1}, \ldots, \varepsilon_{t_{k-1}})$. Therefore, we have to search for multiples $m_{t_{k-1}}$ which have the following form:

$$m_{t_{k-1}} = g(t_{k-1}, Y_{t_1}, \ldots, Y_{t_{k-1}}, \sigma_{t_1}, \ldots, \sigma_{t_k}) \times \mathbb{I}_{C_{t_{k-1}}>0}.$$
In order to determine the conditional multiple, three cases have to be distinguished:

- The cushion at time $t_{k-1}$ satisfies: $C_{t_{k-1}} > L_{t_{k-1}} (> 0)$.
- The cushion at time $t_{k-1}$ satisfies: $0 < C_{t_{k-1}} < L_{t_{k-1}}$.
- The cushion at time $t_{k-1}$ is non positive: $C_{t_{k-1}} < 0$ (in that case, we do no longer invest on the risky asset until the maturity).

Denote:

$$Z_{t_{k-1}} = \alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}} + \sigma_{t_k} \times F^{-1}(1 - (1 - \epsilon)^{1/T}),$$

where $F$ denotes the common cdf of $\epsilon_{t_k}$. For small values of $\epsilon$, $F^{-1}(1 - (1 - \epsilon)^{1/T})$ is negative. It implies that the logreturn value $Z_{t_{k-1}}$ is a decreasing function with respect to the volatility $\sigma_{t_k}$.
Proposition

- **Case 1:** $C_{t_{k-1}} > L_{t_{k-1}}$ :

  - (1-1) If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 + \frac{1}{m_{t_{k-1}}} < 0$, then $\mathbb{P}^{G_{t_{k-1}}}[C_t > L_{t_{k-1}}] = 1$. *If the cushion at time $t_{k-1}$ is relatively high and the multiple is not too high, then portfolio always guaranteed (according to a given criterion).*

  - (1-2) If $\frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 + \frac{1}{m_{t_{k-1}}} > 0$, then:

    - (1-2-i) If $Z_{t_{k-1}} < 0$, the conditional multiple must satisfy:

      $$m_{t_{k-1}} \leq \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \exp[Z_{t_{k-1}}] - 1.$$

    - (1-2-ii) If $Z_{t_{k-1}} > 0$, there is no constraint for $m_{t_{k-1}}$ (recall that we take $m_{t_{k-1}} > 0$).
Quantile and expected shortfall conditions (8)

If the investor wants to use a higher multiple then this one must be smaller than a given upper bound if logreturn value $Z_{t_{k-1}}$ is negative (condition 1-2-i: "bear market"). If the marker is rather bullish (condition 1-2-ii), then the multiple is not bounded.

Proposition

- Case 2: $(0 <) C_{t_{k-1}} < L_{t_{k-1}} : Then$
- (2-i) If $Z_{t_{k-1}} < 0$, there is no solution for $m_{t_{k-1}}$.
- (2-ii) if $Z_{t_{k-1}} > 0$, the conditional multiple must satisfy:

$$m_{t_{k-1}} > \frac{L_{t_{k-1}} - 1}{\exp[Z_{t_{k-1}}] - 1}. \quad (3)$$

Here, since the cushion value at time $t_{k-1}$ is relatively small, the multiple must be sufficiently high in order to get a cushion higher than the threshold at time $t_k$. 
Remark

The previous result shows that, if at time $t_{k-1}$, the auto regressive terms $\alpha_0 + \sum_{i=1}^{P} \alpha_i \times Y_{t_{k-i}}$ is sufficiently high and the conditional volatility $\sigma_{t_k}$ is sufficiently small, then the logreturn $Y_{t_k}$ has a high probability to be positive and thus, there is no constraint on the multiple at time $t_{k-1}$.

Remark

When the cushion is positive at time $t_{k-1}$, the choice of the multiple is very flexible. Thus, within the quantile condition at time $t_{k-1}$, we can add some other conditions on the multiple to better benefit from market conditions. When the cushion is negative (which happens with a small probability), the quantile condition generally cannot be satisfied, except for small values of $(1 - \epsilon)$. But, in this case, this is not a true insurance condition. Therefore, a possible strategy is to adopt the previous condition when the cushion is positive and to invest the whole portfolio value on the riskless asset, as soon as the cushion is negative.
- First, we determine the values of $m_{t_{k-1}}$ which correspond to the quantile condition A3. For the given probability level $\alpha \equiv 1 - (1 - \epsilon) \frac{1}{T}$ in assumption A3), we consider the set $M_{t_{k-1}}(\alpha, L_{t_{k-1}})$ of $m_{t_{k-1}}$ values such that:

$$\mathbb{P}[C_{t_k} \leq L_{t_{k-1}} \mid \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0] \leq \alpha, \; \alpha \epsilon \in \mathbb{R}.$$ 

- Then, the control of the expected shortfall is defined on the previous set from criterion B3:

$$\min_{m_{t_{k-1}} \in M_{t_{k-1}}(\alpha, L_{t_{k-1}})} L_{t_{k-1}} - \mathbb{E}[C_{t_k} \mid \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_{t_k} \leq L_{t_{k-1}}].$$
Denote \( l_L(m_{t_{k-1}}) = \ln \left( \frac{\left( \frac{L_{t_{k-1}}}{C_{t_{k-1}}} - 1 \right)}{m_{t_{k-1}} + 1} \right)^{-a_{t_{k-1}}} \). Then, we deduce:

**Proposition**

*The expected shortfall condition corresponds to the conditional expectation*

\[
\Phi_L(m_{t_{k-1}}) = \mathbb{E}[L_{t_{k-1}} - C_t \mid \mathcal{F}_{t_{k-1}} \cap C_{t_{k-1}} > 0 \cap C_t < L_{t_{k-1}}]
\]

*which is equal to*

\[
L_{t_{k-1}} - \frac{C_{t_{k-1}} \int_{-\infty}^{l_L(m_{t_{k-1}})} \left[ 1 + m_{t_{k-1}} \left( \exp(a_{t_{k-1}} + b_{t_{k-1}}x) - 1 \right) \right] f_{\varepsilon}(x) dx}{\int_{-\infty}^{l_L(m_{t_{k-1}})} f_{\varepsilon}(x) dx}.
\]
Suppose for example that $\varepsilon_{t_{i+1}}$ is Gaussian and $L_{t_{k-1}} = 0$. We have:

$$
\begin{align*}
  l_0(m_{t_{k-1}}) &= \frac{\ln(1 - \frac{1}{m_{t_{k-1}}}) - a_{t_{k-1}}}{b_{t_{k-1}}} , \\
  l_0(m_{t_{k-1}}) - b_{t_{k-1}} &= \frac{\ln(1 - \frac{1}{m_{t_{k-1}}}) - a_{t_{k-1}} - b^2_{t_{k-1}}}{b_{t_{k-1}}} ,
\end{align*}
$$

where $l_0(m_{t_{k-1}})$ does not depend on the cushion value $C_{t_{k-1}}$ at time $t_{k-1}$. Therefore, we get:

$$
\Phi_0(m_{t_{k-1}}) = -C_{t_{k-1}} \left[ 1 - m_{t_{k-1}} \left( 1 - \exp\left( \frac{b^2_{t_{k-1}} + 2a_{t_{k-1}}}{2} \right) \frac{N(l_0(m_{t_{k-1}}) - b_{t_{k-1}})}{N(l_0(m_{t_{k-1}}))} \right) \right] .
$$
Numerical illustrations (1)

- Simulation of the value of the expected shortfall function $\Phi_L$ for several thresholds $L$ and volatilities $\sigma$, for $V_0 = 100$ and $C_0 = 10$.
Numerical illustrations (2)

- The expected shortfall value can reach a (interior) minimum between the lower and upper bounds.
- The multiple value at which the minimum is reached decreases when the volatility increases. It means that the fund manager must reduce the risk exposure when the usual volatility risk rises.
- On the contrary, the multiple value increases when the level floor increases. An interpretation of this feature is that, if the protection due to the floor is more efficient, then we can expose more the remaining fund invested on the risky asset.
- If we choose \( L_{t_{k-1}} < C_{t_{k-1}} \) (\( L = 0.9 \) in previous), the minimization of the expected shortfall \( \Phi_L \left( m_{t_{k-1}} \right) \) gives always a value of the multiple equal to the lower bound. Additionally, for \( L_{t_{k-1}} > C_{t_{k-1}} \), the fund manager must increase the risk exposure in order to expand the probability that \( L_{t_{k-1}} < C_{t_k} \).
We determine the parameters of the EGARCH (1,1) model from the pseudo maximum likelihood method using MATLAB. In what follows, we detail the methodology. Then, we put these parameters into the EGARCH (1,1) model to produce simulations. This method allows to get simulations very close to the true behavior of the S&P 500 historical weekly logreturns on the period 01/1970-11/2011. We have 2217 observations. The sample is split into two periods: the first one (until 11/2006) is used to estimate the parameters; the rest of the sample is used to test the model. Next figure presents the empirical distribution function of the S&P500 weekly logreturns together with the innovations (process $\varepsilon_t$), the conditional standard deviation $\sigma_t$ and the returns. We note that empirical and simulated distributions are very close.
Empirical CDF for simulated and S&P500 weekly returns

Simulated returns
S&P 500 returns

Innovations
Conditional Standard Deviations
Returns


0
0.05
0.1
0.2
Table (1) presents the descriptive statistics of the S&P 500 weekly logreturns on the period 01/1970-11/2011.

<table>
<thead>
<tr>
<th>Mean</th>
<th>maximum</th>
<th>minimum</th>
<th>std deviation</th>
<th>skewness</th>
<th>kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0014</td>
<td>0.1412</td>
<td>−0.1820</td>
<td>0.0229</td>
<td>−0.3184</td>
<td>7.6140</td>
</tr>
</tbody>
</table>

The S&P500 yields present an excess kurtosis compared to the Gaussian distribution (7.6140 > 3). This feature illustrates the fat tails of the probability distribution that shows a significant probability of extreme returns. Note also that the skewness is negative (−0.3184). Therefore, the S&P500 logreturn does not follow a Gaussian distribution. We observe that we have a maximum drop equal to −16%, during the financial crisis.
We use the EGARCH(1,1) model to estimate the conditional volatility. We find:

$$\log \sigma_{t_k}^2 = c_0 + d_1 \log \sigma_{t_{k-1}}^2 + \gamma \left| \epsilon_{t_{k-1}} \right| - \mathbb{E} \left\{ \left| \epsilon_{t_{k-1}} \right| \right\} + \zeta \epsilon_{t_{k-1}},$$

where $\mathbb{E} \left\{ \left| \epsilon_{t_{k-1}} \right| \right\} = \sqrt{\frac{2}{\pi}}$ for the Gaussian law.

Then the conditional quantile $Z_{t_{k-1}}$ is given by:

$$Z_{t_{k-1}} = \alpha_0 + \alpha_1 \times Y_{t_{k-i}} + \sigma_{t_k} \times F^{-1}(1 - (1 - \epsilon)^{1/T}).$$

All terms are statistically significant, in particular the leverage parameter $\zeta$ which is negative. This emphasizes the importance of asymmetrical effect. Next figure shows the paths of $Z_{t_{k-1}}$, those of the conditional multiple determined from the quantile condition and finally those of the conditional volatility. Note that in periods of high volatility, the value of $Z_{t_{k-1}}$ is adjusted upward.
Figure: Paths of Z, of the multiple and of the volatility
Our goal is to determine the multiple value \( m \) that minimizes this function, allowing to reduce the gap risk.

The minimization of the expected shortfall criterion provides multiple values which belong to the interval \([1.5; 4.5]\). These values can be applied in practice. Next figure shows the empirical distribution functions of the payoff of the variable multiples model for \( L = 0 \), for standard multiple for \( m = 3 \) and \( m = 6 \) and for \( T = 5 \) years.

The fixation of a threshold \( L = 0 \) is too restrictive for the model under expected shortfall condition. Certainly, we can have a better guarantee than with standard multiple for \( m = 3 \) and \( m = 6 \), but this guarantee has a cost. It does not take full advantage of market increases.
Numerical illustrations (9)

Figure A: Empirical CPPI cdf for $L = 0$.

Figure B: Empirical CPPI cdf for $L > C_{t_k-1}$.

Figure: CPPI cdf and expected shortfall condition
We apply our model for the very volatile period from 11/2006 to 11/2011.

We adopt a weekly rebalancing.

We compare the result to the standard CPPI model for \( m = 3 \) and \( m = 6 \).

During this period, the market goes through several phases and particularly towards the end where we note the stock market crash related to the subprime crisis.

We suppose that the log-returns of the S&P500 are dependent and follows an \( EGARCH(1,1) \) model.

The selection criteria of this model are outlined in a previous section.
Numerical illustrations (11)

In next figure, the portfolio dynamics for $F_T = 100\% V_0$ and for $F_T = 95\% V_0$ are displayed.
The CPPI model with \( m = 6 \) provides higher gains than other models when the market is bullish, but, with the financial crash of October 2008, the portfolio is fully monetized and falls below the floor at the end of the period for the two floor values \( F_T = 100\% V_0 \) and \( F_T = 95\% V_0 \), although this value of multiple is not considered a very high value. On the other hand, with \( m = 3 \), we were unable to capture large market performance. With the two models of the conditional multiple under VaR condition with a variable threshold \( L_{t_k} = 85\% C_{t_k-1} \) and under the measure risk expected shortfall with \( L = 1.1 C_{t_k-1} \) we can have higher performance than with standard multiple \( m = 3 \). The multiple determined with expected shortfall condition take values in an interval \([1.5; 4.5]\), while the multiple determined with quantile condition take values in an interval \([1.5; 6]\). This allows to explain that portfolio CPPI with VaR condition reaches a higher level than portfolio with expected shortfall condition. Then, with the crisis subprimes and higher volatility on the market, these two models have reduced the exposure to the risky asset S&P500 by decreasing the multiple values.
Conclusion

- As previously shown, it is possible to choose variable multiples for the CPPI method when using quantile hedging, even in the case of dependent logreturns.
- Upper bounds can be calculated for each level of probability and according to state variables.
- This new multiple can be determined according to the distributions of the risky asset logreturn and volatility.
- With the B3 criterion, the investor wants to minimize the expected loss according to the dynamic market information.
- In such a framework, we can determine the adequate value of conditional multiples.
- The performances are significantly different from those of the standard CPPI strategy.
- Other conditions can be imposed on the multiple, while keeping the quantile and expected shortfall hedging constraints.
- Other state variables can also be considered, such as implicit volatility or exogenous financial and macro economic factors.