Stochastic Modeling of Systems Subject to Shocks

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• Systems from simple electrical switches to complicated electronic integrated circuits and from unicellular organisms to human beings are subject to online degradation.
• The result of system ageing is unplanned failure.
• Systems used in the production and servicing sectors which constitute a major share in the industrial capital of any developing nation are subject to online deterioration.
From the industrial perspective the progressive system degradation and failure is often reflected in increased production cost, lower product quality, missed target schedules and extended lead time.

Thus the study of deteriorating systems from the point of view of maintenance and replacement are of paramount importance.
• Early models in such studies dealt with age replacement models. In such models the age of the system was the control variable and the replacement policies called “control limit policies” required to replace the system on reaching a critical age.

• Typical examples are pharmaceutical items, mechanical devices, car batteries, etc. If systems, on failure are replaced with new items, then the failure counting process is a renewal process.
• Minimal Repair: Minimal repairs restore the system to the condition just prior to failure. Thus, if a system fails at age $t$, then the replaced system will have age 0 if the system is replaced by a new one (also called perfect repair), and age $t$ if it is minimally repaired on failure (imperfect repair).

• General Repair: However in practice, system repairs may not correspond to either of these two extreme situations and may result in the repaired system having an age between the age 0 and $t$. Such repairs in maintenance parlance are referred to as general repairs.
The maintenance action mentioned above can be broadly classified as preventive maintenance (PM) and corrective maintenance (CM).

The former is carried out when the systems is working and are generally planned in advance.

PMs are done to improve the reliability of the system.

On the other hand CMs are done on system failure and are unplanned.

Also unplanned maintenance costs more than the planned ones.
Introduction

• All real world systems are deteriorating in nature. Majority of such models approach the problem through shock models. 

  *Abdel Hameed (1986), Feldman (1976), Shanthikumar (1983)*

• A system is subject to randomly occurring shocks, each of which adds a nonnegative random quantity to the accumulated damage process (cumulative damage).
The term shock refers any perturbation to the system.

Shock Models

The system fails when accumulated damages crosses a threshold value

Cumulative damage

series of randomly occurring shocks

Lethal shock

repair time

time

working time

threshold value

δ
Literature Review

• Cox (1962) was the first one to construct stochastic failure models in reliability physics using cumulative processes as well as renewal processes.

• Nakagawa and Osaki (1974) proposed several stochastic failure models for a system subject to shocks.

• The statistical characteristics of interest in their models were the following: (i) the distribution of the total damage (ii) its mean (iii) the distribution of the time to failure of the system (iv) its mean and (v) the failure rate of the system.
• The paper by Taylor (1975) can be considered as a seminal paper on shock models which led to many interesting variations of the shock models.

• He considered the optimal replacement of a system and its additive damage using a compound Poisson process to represent the cumulative damage.
• Nakagawa and Kijima (1989) applied periodic replacement with minimal repair at failure to several cumulative damage models.
• In 1996, Rangan et al considered two optimal stopping problems in a general model of shock associated with a correlated pair of renewal sequences and they denote respectively the time and magnitude of the nth collision.

• Yeh and Zhang [(2002), (2003)] proposed geometric-process maintenance models for deteriorating systems which assumed the shock arrivals to be only independently distributed and not necessarily identically distributed.

• Yeh and Zhang (2004) in a refreshing departure introduced a new class of shock models and called them $\mathcal{S}$-shock models.

• Earlier shock models concentrated solely on the magnitude of the damage caused by the shocks, Yeh and Zhang’s model paid attention to the frequency of the shocks.
Rangan and Tansu (2011): A generalized of a new class of models for renewal shock arrivals and random threshold
**δ-Shock Models**

- A system is likely to fail if two successive shocks occur within a short time, whereas it may not fail if they are separated by a longer duration.

*As an example*
- Think of an elastic material which is subjected to stretching.
- If the second stretching occurs before the material recovers from the first stretching, the material breaks.
The Stochastic Model

- This study attempts a comprehensive analysis of $\delta$-shock models in which the shock counting process is generalized to a renewal process, and

- Threshold times are considered as random variables.

- Apart from deriving explicitly various statistical characteristics of the model we analyze optimal replacement problems of such a system.
Notations Used

• \( Z \): Random variable denoting the time between two successive shocks with pdf \( f_Z(t) \).
• \( D \): Random variable denoting the threshold value with pdf \( g_D(t) \).
• \( W \): Random variable denoting time between two successive failures with pdf \( k_W(t) \).
• \( N(t) \): Counting variable denoting the number of failures in \((0, t)\).
• \( M(t) = E\{N(t)\} \).
• \( L_f(s) \) Laplace Transform of the density function \( f \).
I. The Model Assumptions

Assumption 1

The system is subject to shocks. The time between shocks, are assumed to be independently and identically distributed.
Cycle 1  Cycle 2  Cycle N

Fails  Fails  

Time between two shocks $> \delta$ nonlethal shock
Time between two shocks $< \delta$ lethal shock
Assumption 3

• Threshold value $\delta$ is a random variable.

Assumption 4

• The shock arrival times and the threshold value times are independent of each other.
The Statistical Characteristics

- Probability density of $W$ can be represented as the sum of a random number of random variables,

$$W = \sum_{i=1}^{N-1} X_i + Y_N$$

  time between two successive failures

- $X_i$: a sequence of independently and identically distributed random variables, which are distributed as $Z$ but conditional on $Z > D$.

- $Y_N$: random variable distributed as $Z$ but conditional on $Z \leq D$. It is assumed to be independent of the sequence $X_i$'s.
• N has the geometric distribution

\[ P[N = n] = qp^n, n = 0, 1, 2, \ldots \]

where

\[ q = P[Z \leq D] \]

• Define the conditional distributions of \( X_i \) and \( Y_N \) as

\[ \alpha(x) = P[x < Z < x + dx \mid Z > D] = \frac{f_Z(x)G_D(x)}{P(Z > D)} \]

and

\[ \beta(x) = P[x < Z < x + dx \mid Z \leq D] = \frac{f_Z(x)\bar{G}_D(x)}{P(Z \leq D)} \]
\[ k_w(t) = \sum_{n=0}^{\infty} \left[ \alpha^{(n)} * \beta(t) \right] P(Z \leq D) \left[ P(Z > D) \right]^n \]

• And taking Laplace Transform of \( k_w(t) \):

\[ L_k(s) = \frac{L_{fG}(s)}{1 - L_{fG}(s)} \]

• \( L_k(s) \) could alternatively be derived by writing the integral equation

\[ k_w(t) = f_z(t)G_D(t) + \int_0^t f_z(\tau)G_D(\tau)k_w(t - \tau) d\tau \]
The first shock itself occurs only after $t$

$$P(W > t) \leftarrow \bar{K}_W(t) = F_Z(t) + \int_0^t f_Z(\tau) G_D(\tau) \bar{K}_W(t - \tau) d\tau$$

The first shock occurs at some instant $\tau \in (0, t]$

In the remaining interval $(\tau, t]$ of length $(t - \tau)$ there is no failure

The threshold time starting from $t=0$ is over by time $\tau$
• With simple differentiation;

\[ k_W(t) = f_Z(t)\bar{G}_D(t) + \int_0^t f_Z(\tau)G_D(\tau)k_W(t-\tau)\,d\tau \]

• Application of Laplace Transform we get the

\[ \mathcal{L}_k(s) = \frac{L_f\bar{G}(s)}{1 - L_f\bar{G}(s)} \]

The Mean and Variance are obtained after simplifications:

\[
E[W] = \frac{E(Z)}{P(Z \leq D)}
\]

\[
Var[W] = \frac{E(Z^2)}{P(Z \leq D)} + \frac{2E(Z)E(Z | Z > D)P(Z > D) - E^2(Z)}{P(Z \leq D)^2}
\]
Specific Models and Discussions

• When the system is subjected to the same kind of shock each time, the threshold time of the system is likely to remain a constant, a case discussed by Yeh (2004,2006).

• Under such a scenario, we have considered a few models for different shock arrival distributions:
  – Exponential
  – Gamma
  – Uniform
Conjecture

– For constant threshold times, amongst all the nondegenerate shock arrival distributions with common mean, exponential shock arrivals lead to the minimum mean failure.

– Amongst all the shock arrival distributions with common mean, the degenerate threshold time distribution leads to the minimum mean failure.
II. Simulation and Estimation

- Moment Estimators
- Deteriorating Systems have been widely modeled using Power Law Process which is inhomogeneous Poisson Process with intensity function

\[ \lambda(t) = \frac{\beta}{\theta} \left( \frac{t}{\theta} \right)^{\beta-1}, \quad \beta, \theta, t > 0 \]

- We have fitted a Power Law Process to our model by estimating \( \hat{\beta} \) and \( \hat{\theta} \) from our simulated failure times.

- Hypothesis Testing

- Bayesian Estimation to estimate \( \hat{\beta} \) and \( \hat{\theta} \) for various apriori distributions.
III. Optimal Replacement Model for A Non-Repairable System

• The total cost of running the system

\[ qT + C(t_2 - t_1) + \ldots + N(T)C(T - t_{N(T)}) + c_0 \]

Cost of operating the system/unit time

Period of replacement of the system

Cost of periodic replacement

increase in operating cost rate per unit with successive shocks
• The long run expected cost of operating the system per unit time;

\[ E[C_1(T)] = \frac{aT + C \int_0^T E[N(t)] \, dt + c_0}{T} \]

• The optimal value of the periodic replacement time always exists and is given by the unique solution of the integral equation

\[ \int_0^T [M(T) - M(t)] \, dt = \frac{c_0}{C} \]

where \( M(t) \) is the expected number of failure in \([0, t]\).
• All shock models in the literature have minimized the expected cost per unit time to obtain the optimal $T^*$.  

• By controlling the second order characteristics of the failure process, the performance criterion of the system is controlled, so cost minimization emerges by considering variance of the number of shocks in $(0,T)$. 
• $T_2^*$ be the optimal period for replacement using the cost function

$$E[C_2(T)] = \alpha E[C(T)] + (1 - \alpha)\sqrt{\text{Var}(C(T))}$$

• We have obtained optimal $T_1^*$ and $T_2^*$ for various cases and compared the costs, $C(T_1^*)$ and $C(T_2^*)$. 
IV. Optimal Replacement Model for A Repairable System

During the repair time, system is shut and arriving shocks have no effect on the system.
For our model we have derived $C(N)$, and the Optimal $N^*$ is obtained

Expected successive repair times, form a geometric process with rate $b$

\[
C(N) = \frac{\sum_{n=1}^{N} \frac{1}{b^n} - r \sum_{n=1}^{N} \lambda_n + R}{\sum_{n=1}^{N} \lambda_n + \mu \sum_{n=1}^{N-1} \frac{1}{b^n} + \tau}
\]

Repair cost rate

Replacement cost

Expected Random Replacement time

Expected operating time of the system following the (n-1)th repair in a cycle

• We have derived optimal $N^*$ and the conditions which $N^*$ uniquely exists.
V. Optimal Replacement for A Non-Repairable System and Spare with Lead Time

• This study considers a generalized age-replacement policy of a system subject to shocks with random lead time first considered of Sheu and Chien, 2004.

Model:
1. In the age-replacement policy, planned (scheduled) replacements occur whenever an operating unit reaches age $T$ and a spare unit is available.
2. System failure is governed by our model stated in I so that the distribution function of failure time is $W(t)$.

3. A spare unit for replacement can be delivered upon order and the Lead time $L$ has distribution function $H(t)$. 
The time between system replacements form a cycle. The length of a cycle is calculated using the following 4 mutually exclusive cases.

**Case 1:**

If the ordered spare arrives before time $T$ and no failures occurs before $T$, then delivered unit is put into stock and unit is replaced by that spare at age $T$, at a cost $\eta_1$. 

![Diagram of Case 1](image-url)
Case 2:

If the ordered spare arrives after time $T$ and no failures occurs before the arrival of the ordered spare unit, then the unit is replaced by a spare as soon as the spare is delivered at a cost $\eta_2$.
Case 3:

If the ordered spare arrives before failure which occurs before the time $T$, then the delivered unit is put into stock and the unit replaced by the spare upon the failure at a cost $\eta_3$. 

**Chart:**

- $0$ to $L$:
- $L$ to $T_F$:
- $T_F$ to $T$:
- $L < T_F < T$:
Case 4:

If failure occurs before the arrival of the ordered spare, then the system is shut down and replaced by the spare as soon as the spare is delivered at a cost $\eta_4$

- The cost rate for stocking is $c_s$
- The cost rate resulting from system down is $c_p$
E[cost/unit time] = \frac{E[cost/cycle]}{E[ length of the cycle]}

C(T) = \frac{(\eta_3 - \eta_2) \int_0^T H(t) dW(t) + \eta_2 \int_0^\infty H(t) dW(t) + \eta_4 \int_0^\infty W(t) dH(t) + c_s \int_0^T H(t) \bar{W}(t) dt + c_p \int_0^\infty W(t) \bar{H}(t) dt}{\int_0^\infty \bar{W}(t) H(t) dt + \int_0^\infty \bar{H}(t) dt}

Let

Q(T) = \left[ \int_0^T \bar{W}(t) H(t) dt + \int_0^\infty \bar{H}(t) dt \right] \left[ (\eta_3 - \eta_2) r_w(T) + c_s \right] - (\eta_3 - \eta_2) \int_0^T H(t) w(t) dt - c_s \int_0^T H(t) \bar{W}(t) dt

and

K = \eta_2 \int_0^\infty H(t) w(t) dt + \eta_4 \int_0^\infty W(t) h(t) dt + c_p \int_0^\infty W(t) \bar{H}(t) dt

h(t) : lead time density \quad w(t) : failure time density
• We have established the following:

• \( Q(T) \) is monotonically increasing and

\[
\begin{align*}
Q(0) < K, & \quad Q(\infty) < K & T^* = \infty \\
Q(0) < K, & \quad Q(\infty) > K & T^* = unique, finite \\
Q(0) > K, & \quad Q(\infty) > K & T^* = 0
\end{align*}
\]
VI. Optimal Replacement for A Non-Repairable System with Emergency Replacement

- In system V we considered the case when the system could be shut down due to nonavailability of the spare. However, systems such as power distribution cannot be shut down.

- When the spare is unavailable, we replace the system on failure using emergency replacement. We assume the cost of emergency replacement to be $C_E \gg C_p$. In this case, the expected cost rate is,
\[ C(T) = \frac{(\eta_3 - \eta_2) \int_0^T H(t) dW(t) + \eta_2 \int_0^\infty H(t) dW(t) + \eta_5 \int_0^\infty W(t) dH(t) + c_s \int_0^T H(t) \overline{W}(t) dt}{\int_0^\infty \overline{W}(t) H(t) dt + \int_0^\infty \overline{H}(t) dt} \]

- A similar optimality analysis has been carried out
VII. Non-Repairable System and Spare with no Lead Time

\[
C(T) = \frac{c_1 \overline{W}(T) + c_2 \overline{W}(T)}{\int_0^T \overline{W}(t) \, dt}
\]

and

\[
C'(T) = \frac{\overline{W}(T) \left[ (c_1 - c_2) \{ r_w(T) \int_0^T \overline{W}(t) \, dt + \overline{W}(T) \} - c_1 \right]}{\left( \int_0^T \overline{W}(T) \, dt \right)^2}
\]
Optimality Conditions

1. If $c_1 \leq c_2$, $C(T)$ is a decreasing function, then $T^* = \infty$.

2. If $c_1 > c_2$,
   
   Let 
   
   $$B(T) = (c_1 - c_2) \{ r_w(T) \int_0^T \overline{W}(t) dt + \overline{W}(T) \} - c_1$$

   The optimal $T^*$ is given as the solution of $B(T^*) = 0$

   If $B(T) = 0$ has no solution then $T^* = \infty$

   Further if $r_w(T)$ is increasing which is a natural assumption, it has been shown that $B(T)$ is an increasing function, establishing the uniqueness of $T^*$. 
Optimality Conditions

• The condition for the existence of a finite optimal $T^*$, 

$$ r_{w}(\infty) > \frac{c_1}{(c_1 - c_2)E[W]} $$
Application Areas

• One of the major reasons for employing the shock model approach is that they have wide applicability in other areas as well. We list below a few of them:

• 1. Cumulative damage models: These models could be applied to stochastic clearing systems, fatigue failure of materials subjected to cyclical and random loading, inventory control and Queuing theory.
Application Areas

2. Optimal stopping models: Cancer chemotherapy, handling of toxic materials in chemical industries and metal fatigue in devices.

3. Failure models based on frequency of shocks: Renewing warranty models, reliability of one unit system with the single spare and one repair facility, studies in single neurons.


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– “Some Results on a New Class of Shock Models”
Asia-Pacific Journal of Operational Research (APJOR) 2011
Thank You